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ALMOST TOPOLOGICAL DYNAMICAL SYSTEMS

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ABSTRACT
For the notion of finitary isomorphism, which arises in many examples in ergodic theory, we prove some basic theorems about invariants, representations and the central limit theorem in shift spaces.

Flows arising in physics or ergodic theory often present singular behavior on a negligible set of states, although their topological properties are important. Almost topological dynamical systems form a suitable model for describing, both topologically and metrically, asymptotic comportment of these flows. They are defined as measure-preserving transformations acting on a compact metric space which are continuous after removal of a negligible set of singularities. Much attention in recent years has been paid in ergodic theory to the interplay of topological and metric properties (see [10], [18]), and we attempt here to present the foundations for a systematic study.

The first section contains the basic definitions of almost topological dynamical systems and finitary homomorphisms and isomorphisms of such systems. Descriptions of invariants for finitary isomorphisms are contained in the second paragraph, the most important of which is strict ergodicity (Theorem 7). From this it follows that metrical isomorphism classes are in general larger than finitary classes, in particular in the case of Bernoulli schemes.

In the third section we show that any aperiodic almost topological dynamical system is finitarily isomorphic to a shift dynamical system, and in the ergodic case with finite entropy the existence of a finite (almost topological) generator with the minimal number of symbols is derived.

The last section deals with finitary homomorphisms between two shift dynamical systems. Such homomorphisms can be effected by a sequential coding procedure, and we look at those homomorphisms for which the coding has a finite expectation (Definition 22). In this case, we show that if the coordinate process of a dynamical system satisfies a certain mixing condition, then any

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sufficiently smooth function on the homomorphic image space satisfies the central limit theorem.

§1. Almost topological dynamical systems

Let Ω be a compact metric space, m a probability measure on the Borel subsets of Ω, and denote by $\mathcal{F}$ the completion of the Borel σ-field with respect to m.

DEFINITION 1. An *almost topological dynamical system* on $(\Omega, \mathcal{F}, m)$ is given by a subset Ω_1 of Ω and a map $T : \Omega_1 \to \Omega_1$ such that:

AT1. Ω_1 is a residual subset of Ω.

AT2. T is a homeomorphism of Ω_1 (with the induced topology).

AT3. $\Omega_1 \in \mathcal{F}$ and $m(\Omega_1) = 1$.

AT4. T preserves the probability measure m (restricted to Ω_1).

For convenience we have restricted attention to invertible systems, and the generalization to non-invertible systems presents little difficulty. In the sequel, "dynamical system" will always mean an almost topological dynamical system in the sense of Definition 1. A *topological dynamical system* will be for our purposes a dynamical system for which $\Omega_1 = \Omega$, whereas a *metrical* (measure theoretical) *dynamical system* will designate, as usual, an automorphism of a Lebesgue probability space. Note that a topological dynamical system can be considered to be an (almost topological) dynamical system, and that an (almost topological) dynamical system gives rise to a metrical dynamical system by neglecting the topology.

DEFINITION 2. Let T and T' be dynamical systems on $(\Omega, \mathcal{F}, m)$ and $(\Omega', \mathcal{F}', m')$ respectively, with domains of definition Ω_1 and Ω'_1. An *almost topological* or *finitary* homomorphism from T to T' is given by subsets $\Omega_2 \subseteq \Omega_1$, $\Omega'_2 \subseteq \Omega'_1$ and a map $\varphi : \Omega_2 \to \Omega'_2$ such that:

ATH1. Ω_2 and Ω'_2 are residual subsets of Ω and Ω'.

ATH2. φ is continuous and surjective.

ATH3. $\Omega_2 \in \mathcal{F}$, $\Omega'_2 \in \mathcal{F}'$ and $m(\Omega_2) = m'(\Omega'_2) = 1$.

ATH4. φ maps m to m'.

ATH5. $\varphi T = T'\varphi$ on Ω_2.

In addition, φ is called a *finitary isomorphism* if φ is invertible and φ^{-1} is continuous.

We remark that it is equivalent to require that Ω_2 and Ω_2' be residual in Ω_1 and Ω_1' respectively, and that we may assume $\Omega_2 = T\Omega_2$ and $\Omega_2' = T'\Omega_2'$ (by replacing Ω_2 by $\bigcap_{k\in\mathbf{Z}} T^k\Omega_2$ and correspondingly for Ω_2'). Moreover, if T and T' are dynamical systems on the same space $(\Omega, \mathcal{F}, m)$ with domains of definition Ω_1 and Ω_1' respectively, and if T and T' agree on a residual subset of $\Omega_1 \cap \Omega_1'$, then T and T' are finitarily isomorphic. Any T-invariant residual subset of full measure will be called a *carrier* for T, and obviously Ω_1 can be replaced by any carrier for T without changing the isomorphism class. A countable intersection of carriers for T is again a carrier for T. Finitary isomorphism is obviously an equivalence relation on the category of almost topological dynamical systems.

THEOREM 3. *Let $(\Omega, \mathcal{F}, m, T)$ be a dynamical system. Then there exists a topological dynamical system $(\Omega', \mathcal{F}', m', T')$ which is finitarily isomorphic to $(\Omega, \mathcal{F}, m, T)$.*

PROOF. Let Ω_1 be a carrier for T. If d denotes the (bounded) metric on Ω, define a new metric d_1 on Ω_1 by setting

$$d_1(x, y) = \sum_{k\in\mathbf{Z}} 2^{-|k|} d(T^k x, T^k y).$$

Then $d \leqq d_1$, and if $x_n, x \in \Omega_1$ with $\lim_n d(x_n, x) = 0$, it follows that $\lim_n d(T^k x_n, T^k x) = 0$ (since T is continuous on Ω_1), and thus $\lim_n d_1(x_n, x) = 0$. Hence d and d_1 define the same topology on Ω_1. Moreover,

$$\tfrac{1}{2} d_1(x, y) \leqq d_1(Tx, Ty) \leqq 2 d_1(x, y) \qquad (x, y \in \Omega_1),$$

so that T and T^{-1} are uniformly continuous with respect to d_1. For any $\varepsilon > 0$, choose k_0 with

$$\sum_{|k|\geqq k_0} 2^{-|k|} \sup_{x,y\in\Omega_1} d(x, y) < \varepsilon/2.$$

Now the metric $\Sigma_{|k|<k_0} 2^{-|k|} d(T^k x, T^k y)$ satisfies the total boundedness condition, so that there exist $x_1, \cdots, x_n$ with

$$\Omega_1 \subset \bigcup_{i=1}^{n} \left\{ y : \sum_{|k|<k_0} 2^{-|k|} d(T^k x_i, T^k y) < \frac{\varepsilon}{2} \right\}.$$

It follows that

$$\Omega_1 = \bigcup_{i=1}^{n} \{ y : d_1(x_i, y) < \varepsilon \}$$

and hence d_1 is totally bounded on Ω_1. Denote the completion of Ω_1 with respect to d_1 by Ω'. Then Ω' is a compact metric space, and $T:\Omega_1\to\Omega_1$, being uniformly continuous, can be extended to a homeomorphism $T':\Omega'\to\Omega'$. Since $d\leqq d_1$, there is a continuous map $\varphi:\Omega'\to\Omega$ which extends the identity map on Ω_1, and this shows that $\varphi^{-1}(\Omega_1)$ is residual in Ω', since Ω_1 is residual in Ω and $\varphi^{-1}(\Omega_1)$ dense in Ω'. Defining m on Ω' to be 0 on $\Omega'\backslash\Omega_1$ and to agree with m on Ω_1, we obtain the desired result.

Almost topological dynamical systems and finitary homomorphisms between such systems appear frequently in the literature. Since we discussed some examples in [10] and [18] we prefer to give the references to papers where finitary isomorphisms of dynamical systems occur, without claiming to list all of them: [1], [2], [3], [4], [6], [7], [8], [17], [19], [20], [25], and [27].

§2. Invariants

We begin our study of invariants by stating the following proposition, whose proof is obvious in view of the remarks following the definitions of the first paragraph.

PROPOSITION 4. *Any invariant (under metrical isomorphism) defined for metrical dynamical systems is also an invariant (under finitary isomorphism) for almost topological dynamical systems.*

Such invariants are, for example, aperiodicity, ergodicity, weak mixing, strong mixing, K-system, Bernoullicity, etc.

In the case of topological properties, the situation is somewhat different. If $(\Omega,\mathcal{F},m,T)$ is an almost topological dynamical system and if Ω_1 is the domain of definition of T, then T is a homeomorphism of the Baire metric space Ω_1, and any property of such homeomorphisms is then defined for $(\Omega,\mathcal{F},m,T)$. Not all such properties will be invariant under finitary isomorphisms, since such an isomorphism is only defined on a residual subset of Ω_1. We obtain thus the following remark.

PROPOSITION 5. *Any property of homeomorphisms of Baire metric spaces which persists after removal of an invariant set of first category is a finitary isomorphism invariant for almost topological dynamical systems.*

Here are some examples for properties of this type. Let X be a Baire metric space and S a homeomorphism of X.

(a) *Topological transitivity.* S is topologically transitive if there exists a point in X with a dense S-orbit. It follows (see e.g. [28], theor. 5.4, p. 117) that the set of points with dense orbits is a residual subset of X, so that topological transitivity satisfies the condition of Proposition 5.

(b) *Minimal almost periodicity.* S is minimally almost periodic if there exists a point $x_0 \in X$ whose orbit is dense and which is almost periodic, i.e. for any open set U with $x \in U$, there exists an integer N such that

$$\mathbf{Z} = \{k + j : T^k x_0 \in U, 0 \leqq j \leqq N, k, j \in \mathbf{Z}\}.$$

If S is minimally almost periodic, then any point of X has these properties ([13], theor. 4.34, p. 37), so that minimal almost periodicity is a finitary invariant. Note that the situation here differs from the compact case, i.e. it is not sufficient to require that all orbits are dense for minimal almost periodicity, and that minimality in this sense is not an invariant.

(c) *Regional recursivity.* S is regionally recursive if every point $x \in X$ is non-wandering (i.e. for any open U containing x, there exists $n \neq 0$ with $S^n U \cap U \neq \emptyset$). Since the set of non-wandering points is closed (see [13], theor. 3.26, p. 24), regional recursivity is a finitary invariant.

(d) *Topological strong mixing.* S is topologically strongly mixing if for any non-empty open sets U and V, the intersection $U \cap S^n V$ is non-empty for all sufficiently large n. As $U \cap S^n V$ is open, it will remain non-empty after a first category change, and this concept is also finitarily invariant.

(e) *Regular strong mixing.* S is regularly strongly mixing if for any non-empty open set U there exists a positive integer d such that for all n,

$$U \cap S^d U \cap S^{2d} U \cap \cdots \cap S^{nd} U \neq \emptyset.$$

This is a finitary invariant for the same reason as in (d).

There are many more possibilities for topological invariants. The choices alone serve to illustrate the general method. It would be interesting to investigate whether or to which degree other well-known topological properties are preserved by finitary isomorphism, such as distality, specification, topological entropy, or properties of the set of all invariant probability measures. It is probable that one needs to require additional conditions on the isomorphism in these cases. (For topological entropy see e.g. [11].)

Our next concept is a mixture of the topological and measure theoretic aspects of the dynamical systems.

DEFINITION 6. The dynamical system $(\Omega, \mathcal{F}, m, T)$ is said to be *strictly ergodic* if it is minimally almost periodic and if there exists a point $x_0 \in \Omega_1$ which is strictly transitive, i.e. for any $f \in C(\Omega)$, the ergodic averages

$$\frac{1}{n}\sum_{i=k}^{k+n-1} f(T^i x_0)$$

converge, as n tends to ∞ and uniformly in $k \in \mathbf{Z}$, to $\int f dm$.

It is easily shown (using the same argument as in [9], prop. 5.15, p. 28) that if $(\Omega, \mathcal{F}, m, T)$ is strictly ergodic, then every point of Ω_1 is strictly transitive, and that in the definition above the functions $f \in C(\Omega)$ can be replaced by characteristic functions of open sets with boundary measure zero, yielding an equivalent definition. We denote by $\mathcal{U}_0$ the algebra of open subsets of Ω with boundary measure zero.

THEOREM 7. *Let $(\Omega, \mathcal{F}, m, T)$ and $(\Omega', \mathcal{F}', m', T')$ be dynamical systems, and let φ be a homomorphism from T to T'. Then*

(i) *for any $U' \in \mathcal{U}'_0$ there exists $U \in \mathcal{U}_0$ such that*

$$\varphi^{-1}(U' \cap \Omega'_2) = U \cap \Omega_2,$$

and

(ii) *if $(\Omega, \mathcal{F}, m, T)$ is strictly ergodic, then $(\Omega', \mathcal{F}', m', T')$ is strictly ergodic.*

PROOF. Let $U' \in \mathcal{U}'_0$. Since φ is continuous and $U' \cap \Omega'_2$ is relatively open in Ω'_2, $\varphi^{-1}(U' \cap \Omega'_2)$ is relatively open in Ω_2 and there exists U open in Ω with

$$\varphi^{-1}(U' \cap \Omega'_2) = U \cap \Omega_2.$$

Suppose that $x \in \partial U \cap \Omega_2$, and set $x' = \varphi(x)$. Then $x \notin U$, $x' \notin U'$, and there exists a sequence $x_n \in U$ with $\lim_{n\to\infty} x_n = x$. Since U is open and Ω_2 is dense, we may assume also that $x_n \in \Omega_2$ for each n. It follows that $x'_n = \varphi(x_n) \in U'$ and $\lim_{n\to\infty} x'_n = x'$. Hence $x' \in \partial U'$ and

$$\partial U \cap \Omega_2 \subseteq \varphi^{-1}(\partial U' \cap \Omega'_2).$$

Therefore

$$m(\partial U) = m(\partial U \cap \Omega_2) \leqq m'(\partial U' \cap \Omega'_2) = 0$$

and $U \in \mathcal{U}_0$. This proves (i), and (ii) follows immediately from (i), since if U and U' are as above, and if $x \in \Omega_2$ with $x' = \varphi(x)$, then for all k and n

$$\frac{1}{n}\sum_{i=k}^{k+n-1} 1_U(T^ix) = \frac{1}{n}\sum_{i=k}^{k+n-1} 1_{U'}(T^ix'),$$

which implies that x' is strictly transitive if x is.

H. Furstenberg pointed out to us that our hypothesis, that the residual sets of the dynamical systems have full measure, is necessary, since there are simple examples of almost 1–1 extensions of equicontinuous flows which are not strictly ergodic.

COROLLARY 8. *Strict ergodicity is a finitary isomorphism invariant.*

COROLLARY 9. *A dynamical system is strictly ergodic if and only if it is finitarily isomorphic to a strictly ergodic topological dynamical system.*

PROOF. Use Theorem 3.

COROLLARY 10. *Let $(\Omega, \mathcal{F}, m, T)$ be an ergodic dynamical system which is not strictly ergodic. Then there exists a dynamical system $(\Omega', \mathcal{F}', m', T')$ such that T and T' are metrically isomorphic but not finitarily isomorphic.*

PROOF. Use the strictly ergodic embedding theorem ([16], [23], [9]) to obtain $(\Omega', \mathcal{F}', m', T')$ strictly ergodic with T and T' metrically isomorphic. T and T' are not finitarily isomorphic because of Corollary 8.

It is possible that if $(\Omega, \mathcal{F}, m)$ is non-atomic and T is ergodic, then the finitary isomorphism class of T is always strictly smaller than the metrical isomorphism class. We do not know whether this is true, say, for an irrational rotation of the circle.

An obvious generalization of Corollary 10 yields the existence of a Bernoulli transformation (strictly ergodic) which is not finitarily isomorphic to any Markov process, and even has no finitary factors which are Markov processes. Another way (due to K. Dehnad) is the following:

LEMMA 11. *Let $(\Omega, \mathcal{F}, m, T)$ be an aperiodic dynamical system. There exists a set $A \in \mathcal{F}$ with $m(A) > 0$ which does not satisfy the regular strong mixing condition, i.e. for every positive integer d there exists an $n = n(d)$ such that*

$$A \cap T^dA \cap \cdots \cap T^{n(d)d}A = \emptyset.$$

PROOF. Apply Rohlin's lemma ([9], p. 7) to the aperiodic transformation T^d

and the pair $(n_d, 1/n_d)$, where we choose n_d later, to obtain a set B_d such that $T^{td}B_d$, $0 \leqq t < n_d$ are disjoint and

$$m\left(\Omega \setminus \bigcup_{t=0}^{n_d-1} T^{td}B_d\right) < \frac{1}{n_d}.$$

Set

$$A_d = \sum_{t=1}^{n_d-2} T^{td}B_d.$$

Then $m(A_d) \geqq 1 - 2/n_d$ and $\bigcap_{t=0}^{n_d-1} T^{td}A_d = \emptyset$.

Now define

$$A = \bigcap_{d=1}^{\infty} A_d,$$

and choose n_d with $\Sigma_{d=1}^{\infty} 2/n_d < 1$. We have then $m(A) > 0$ and A has the required property.

Now let $(\Omega, \mathcal{F}, m, T)$ be a Bernoulli scheme and A a set as in Lemma 11. Define $\Omega' = \{0, 1\}^{\mathbf{Z}}$, T' the shift on Ω', and m' the image of m under the mapping $\varphi : \Omega \to \Omega'$ induced by the partition $\{A, X \setminus A\}$ of Ω. Then $(\Omega', \mathcal{F}', m', T')$ is a Bernoulli transformation and the cylinder set

$$[0] = \{\omega' \in \Omega' : \omega'_0 = 0\} = \varphi(A)$$

does not satisfy the regular strong mixing condition. Since any open subset of a Markov process satisfies this condition, and since a finitary homomorphism ψ from a Markov process to $(\Omega', \mathcal{F}', m', T')$ carries the open set $[0] \in \mathcal{F}'$ to an open set $\psi^{-1}([0])$ in the Markov process, such a homomorphism cannot exist. Putting everything together, we obtain:

THEOREM 12. *There exist dynamical systems* $(\Omega, \mathcal{F}, m, T)$ *and* $(\Omega', \mathcal{F}', m', T')$, *both metrically isomorphic to Bernoulli schemes, such that no Markov process is a finitary factor of* $(\Omega, \mathcal{F}, m, T)$, *and such that* $(\Omega', \mathcal{F}', m', T')$ *is not a finitary factor of any Markov process.*

In particular, these systems are not finitarily isomorphic to Bernoulli schemes or Markov processes. We do not know whether Theorem 12 can be realized with the same dynamical system.

§3. Generators

All measures used in this section will be assumed to be positive on open sets. We denote by A either a finite set $\{1, \cdots, a\}$ or the one-point compactification $\{0, 1, 2, \cdots, \infty\}$ of $\mathbf{N}$. The product space $X = A^{\mathbf{Z}}$ is compact and metrizable, and the (left) shift transformation S is a homeomorphism of X.

DEFINITION 14. A *shift dynamical system* is a topological dynamical system $(Y, \mathcal{B}, \nu, S)$, where Y is a closed S-invariant subset of X, ν is an S-invariant probability measure on Y, and if A is infinite, $\nu(\mathbf{N}^{\mathbf{Z}} \cap Y) = 1$.

DEFINITION 15. Let $(\Omega, \mathcal{F}, m, T)$ be a dynamical system. A *partition* of $(\Omega, \mathcal{F}, m, T)$ is a finite or countable sequence $\alpha = (U_0, U_1, \cdots)$ of open subsets of Ω such that

$$U_i \cap U_j = \varnothing \qquad \text{if } i \neq j,$$

$$m(\partial U_i) = 0 \qquad \text{for all } i,$$

$$m\left(\bigcup U_i\right) = 1,$$

$$\Omega \doteq \overline{\bigcup_i U_i}.$$

If $\alpha = (U_0, U_1, \cdots)$ is a partition of $(\Omega, \mathcal{F}, m, T)$ and if Ω_1 is a carrier for T, we set

$$\Omega_\alpha = \bigcap_{n \in \mathbf{Z}} T^n\Big(\bigcup_i U_i \cap \Omega_1\Big).$$

Then Ω_α is a residual subset of full measure in Ω, and if A denotes the index set of α (or its compactification if it is infinite), then we can define a map

$$\varphi_\alpha : \Omega_\alpha \to X = A^{\mathbf{Z}}$$

by setting

$$\varphi_\alpha(\omega) = x = (\cdots, x_{-1}, x_0, x_1, \cdots)$$

if $T^n\omega \in U_{x_n}$ for each $n \in \mathbf{Z}$. Defining $Y_\alpha = \overline{\varphi_\alpha(\Omega_\alpha)}$ and $\nu_\alpha = \varphi_\alpha(m)$, we obtain a shift dynamical system $(Y_\alpha, \mathcal{B}, \nu_\alpha, S)$, and since $\varphi_\alpha(\Omega_\alpha)$ is residual in Y_α (this is a nice exercise, which we leave to the reader), φ_α is a finitary homomorphism from $(\Omega, \mathcal{F}, m, T)$ to $(Y_\alpha, \mathcal{B}, \nu_\alpha, S)$.

DEFINITION 16. The partition α is called a *generator* if there exists a subset $\Omega_2 \subseteq \Omega_\alpha$ such that the restriction of φ_α to Ω_2 is a finitary isomorphism. A generator α is *finite* if its index is finite.

Conversely, any finitary homomorphism φ from a dynamical system $(\Omega, \mathcal{F}, m, T)$ to a shift dynamical system $(Y, \mathcal{B}, \nu, S)$ yields a partition $\alpha = (U_i)$ of $(\Omega, \mathcal{F}, m, T)$, namely by choosing for each i an open set U_i such that

$$U_i \cap \Omega_2 = \varphi^{-1}(\{y \in Y : y_0 = i\} \cap Y_2),$$

and for this partition we obviously have $\varphi = \varphi_\alpha$ on $\Omega_2 \cap \Omega_\alpha$. If φ was a finitary isomorphism, then α will be a generator.

The following proposition gives necessary and sufficient conditions for a partition to be a generator.

PROPOSITION 17. *Let $\alpha = (U_i)$ be a partition of $(\Omega, \mathcal{F}, m, T)$. Then α is a generator if and only if there exists a set $\Omega_2 \subseteq \Omega_\alpha$ residual and of full measure for which one of the following equivalent conditions is satisfied:*

(i) *For any $\omega \in \Omega_2$,*

$$\Omega_2 \cap \bigcap_{k=-n}^{n} T^k U_{y_k}, \qquad n = 0, 1, 2, \cdots$$

is an open subbasis for ω in Ω_2 (where $y = \varphi_\alpha(\omega)$).

(ii) *For any $\omega \in \Omega_2$,*

$$\bigcap_{k=-n}^{n} T^k U_{y_k}, \qquad n = 0, 1, 2, \cdots$$

is an open subbasis for ω in Ω.

(iii) *For any $\omega \in \Omega_2$,*

$$\{\omega\} = \bigcap_{k \in \mathbf{Z}} T^k \overline{U_{y_k}}.$$

(iv) *For any $\omega \in \Omega_2$,*

$$\lim_{n \to \infty} \operatorname{diam} \bigcap_{k=-n}^{n} T^k U_{y_k} = 0.$$

PROOF. A sequence $V_1, V_2, \cdots$ of open sets containing a point x in a compact metric space is a subbasis for x if and only if $\bigcap_k \overline{V_k} = \{x\}$, or equivalently $\lim_{k \to \infty} \operatorname{diam} V_k = 0$.

This shows that (ii), (iii) and (iv) are equivalent. Moreover, (ii) obviously implies (i), and (i) implies (ii) because Ω_2 is dense in Ω. Thus (i)–(iv) are equivalent. Now if α is a generator and $\varphi : \Omega_2 \to \varphi_\alpha(\Omega_2)$ is the finitary isomorphism, then the sets in (i) are just the inverse images of the $(-n, +n)$ cylinder sets in Y which form an open subbasis for y. Since φ is a homeomorphism, (ii) follows. Conversely, if (i)–(iv) are satisfied, then (iii) shows that $\varphi_\alpha|_{\Omega_2}$ is injective, and (ii) implies that φ_α^{-1} when restricted to $\varphi_\alpha(\Omega_2)$ is continuous. Thus φ_α is a finitary isomorphism.

THEOREM 18. *Any aperiodic dynamical system has a generator.*

PROOF. Let $(\Omega, \mathcal{F}, m, T)$ be aperiodic. Our first step is to prove that Rohlin's lemma can be applied in a topological setting.

(1) For any $\varepsilon > 0$ and $n \geqq 1$ there exists an open set U in Ω_1 with $U, TU, \cdots, T^{n-1}U$ pairwise disjoint, $m(\partial U) = 0$ and

$$m\left(\Omega \setminus \sum_{i=0}^{n-1} T^i U\right) < \varepsilon.$$

We may of course assume that T is a homeomorphism of Ω (Theorem 3). Rohlin's lemma ([9]) says that we can find a measurable set F with the desired properties. Since any measurable set can be approximated from below by a compact set and then the compact set from above by an open set, we can thus find an open set U with $m(\partial U) = 0$ arbitrarily close to F and such that $U, TU, \cdots, T^{n-1}U$ are disjoint. This shows that (1) is valid.

(2) There exists a partition $\alpha = (U_i)$ of Ω and a sequence $\varepsilon_i \to 0$ such that for each $i = 0, 1, 2, \cdots, U_i \neq \varnothing$ and

$$m\left(\bigcup_{t \in \mathbf{Z}} T^t U_i\right) > 1 - \varepsilon_i.$$

The construction of α will be done by induction. Let $\varepsilon_i \leqq \frac{1}{4}$ be any sequence tending to zero. Choose U_1' as in (1) for $n = 3$ and ε_1. Then we set

$$U_1 = U_1' \cup TU_1'$$

and

$$\Sigma_1 = \Omega \setminus \bar{U}_1.$$

Obviously

$$m\left(\bigcup_{t \in \mathbf{Z}} T^t U_1\right) > 1 - \varepsilon_1$$

and

$$(*) \qquad m\Big(\bigcup_{t\in\mathbf{Z}} T^t\Sigma_1\Big)=1.$$

To obtain U_2 (and Σ_2), let T_1 be the induced transformation from T on the open set Σ_1. It is easy to see that T_1 is continuous when restricted to

$$\Sigma_1\setminus \bigcup_{t\in\mathbf{Z}} T^t(\partial\Sigma_1),$$

so that T_1 is almost topological on Σ_1. Now choose U_2' as in (1) for $n=3$ and ε_2', as a subset of (Σ_1, T_1). We set

$$U_2=U_2'\cup T_1U_2'$$

and

$$\Sigma_2=\Sigma_1\setminus\overline{U_2}.$$

Then $(*)$ implies that

$$m\Big(\bigcup_{t\in\mathbf{Z}} T^tU_2\Big)>1-\varepsilon_2$$

if ε_2' is sufficiently small, and

$$m\Big(\bigcup_{t\in\mathbf{Z}} T^t\Sigma_2\Big)=1.$$

Continuing in this fashion, we obtain a sequence (U_i) of pairwise disjoint open sets with the desired property, and since

$$m(\Sigma_{i+1})\leqq \tfrac{1}{2}m(\Sigma_i),$$

(U_i) is a partition.

(3) Now let $\alpha_i=(U_j^i)$ be a sequence of finite partitions with $\lim_i \sup_j \operatorname{diam}(U_j^i)=0$. Choose n_i with

$$m\Big(\bigcup_{t=-n_i}^{n_i} T^tU_i\Big)>1-2\varepsilon_i,$$

and define the partition β to be $\bigvee_{t=-n_i}^{n_i} T^t\alpha_i$ on the set U_i, $i=0,1,\cdots$. If we set

$$\Omega_2=\bigcap_k\ \bigcup_{i\geqq k}\ \bigcup_{t=-n_i}^{n_i} T^tU_i,$$

then Ω_2 is residual, $m(\Omega_2)=1$, and for any $\omega \in \Omega_2$, condition (iv) of Proposition 17 follows from the construction. Therefore β is a generator.

Our next theorem shows that in most cases, the requirement that a finitary isomorphism be defined on a residual set is automatically fulfilled.

THEOREM 19. *Let* $(\Omega, \mathcal{F}, m, T)$ *and* $(\Omega', \mathcal{F}', m', T')$ *be aperiodic dynamical systems,* $\Omega_2 \subseteq \Omega_1$, $\Omega_2' \subseteq \Omega_1'$, *and* $\varphi : \Omega_2 \to \Omega_2'$ *be such that*:

(i) φ *is a homeomorphism, and* Ω_2 *and* Ω_2' *are dense in* Ω *and* Ω'.

(ii) $\Omega_2 \in \mathcal{F}$, $\Omega_2' \in \mathcal{F}'$ *and* $m(\Omega_2) = m'(\Omega_2') = 1$.

(iii) $\varphi m = m'$.

(iv) $\varphi T = T' \varphi$ *on* Ω_2.

Then $(\Omega, \mathcal{F}, m, T)$ *and* $(\Omega', \mathcal{F}', m', T')$ *are finitarily isomorphic.*

PROOF. According to Theorem 18, we may assume that Ω and Ω' are shift dynamical systems. We want to show that $\varphi : \Omega_2 \to \Omega_2'$ can be extended to a homeomorphism $\varphi : \Omega_3 \to \Omega_3'$ with Ω_3 residual, Ω_3' residual, and $\varphi T = T'\varphi$ on Ω_3. For any cylinder

$$[i] = \{\omega' : \omega_0' = i\}$$

consider

$$\varphi^{-1}([i] \cap \Omega_2').$$

Since this set is open in Ω_2, there exists an open set U_i in Ω with

$$U_i \cap \Omega_2 = \varphi^{-1}([i] \cap \Omega_2').$$

Then $\alpha = (U_i)$ is a partition of Ω, and φ_α maps the residual set $\Omega_\alpha = \bigcap_{n \in \mathbf{Z}} T^n(\bigcup_i U_i \cap \Omega_1)$ into Ω' (here Ω_1 is the domain of definition of T). Similarly, by starting from Ω' and φ^{-1}, we can construct a partition $\alpha' = (U_i')$ of Ω' and a map $\varphi_{\alpha'} : \Omega_{\alpha'}' \to \Omega$. By definition, φ_α and φ agree on Ω_2, and $\varphi_{\alpha'}$ and φ^{-1} agree on Ω_2'. Therefore $\varphi_{\alpha'} \circ \varphi_\alpha$ is the identity map on Ω_2, and since Ω_2 is dense in Ω, $\varphi_{\alpha'} \circ \varphi_\alpha$ is the identity map on $\Omega_\alpha \cap \varphi_\alpha^{-1}(\Omega_{\alpha'}')$. Since φ_α and $\varphi_{\alpha'}$ are continuous, we may set $\Omega_3 = \Omega_\alpha \cap \varphi_\alpha^{-1}(\Omega_{\alpha'}')$, $\Omega_3' = \varphi_\alpha(\Omega_3)$, and then $\varphi_\alpha : \Omega_3 \to \Omega_3'$ is our desired isomorphism, as Ω_3 and Ω_3' are obviously residual.

We now examine the existence of finite generators.

THEOREM 20. *Let* $(\Omega, \mathcal{F}, m, T)$ *be an ergodic dynamical system with entropy* $h(T) < \infty$. *Then there exists a finite generator for* $(\Omega, \mathcal{F}, m, T)$ *whose index set has cardinality* $\leqq e^{h(T)} + 1$.

PROOF. We may assume that T is aperiodic and, for simplicity, that T is a homeomorphism on Ω (see Theorem 3). Since the proof will follow the lines of the well known techniques of proving a finite generator theorem (cf. [9], p. 282 ff.), we leave the details as an exercise.

(1) What has been shown in the proof of Theorem 18 (part (1)) is also one of the essential tools for the construction of a finite (almost topological) generator. For any $\varepsilon > 0$ and $n \in \mathbf{N}$ there exists an open Rohlin set U in Ω such that $U, TU, \cdots, T^{n-1}U$ are disjoint, $m(\partial U) = 0$ and $m(U \cup \cdots \cup T^{n-1}U) > 1 - \varepsilon$. Moreover using the same approximation argument it follows that Grillenberger's lemma ([9], lemma 26.4, p. 256) holds in a topological setting also: Let $\Sigma \subset \Omega$ be open, $m(\partial\Sigma) = 0$ and $0 < m(\Sigma) < 1$. Let T_Σ denote the induced transformation of T on Σ. If $0 < \varepsilon$, $\delta < 1$, then for all sufficiently large $n \in \mathbf{N}$ there exists a $k_n > nm(\Sigma)$ such that for any open set $V \subseteq \Sigma$ with $m(V) > (1-\delta)m(\Sigma)$ and $m(\partial V) = 0$ there exists an open set $U \subseteq V$ with $m(\partial U) = 0$ satisfying

(i) $U, T_\Sigma U, \cdots, T_\Sigma^{k_n - 1}U$ are disjoint and cover all but a set of measure $< (\varepsilon + \delta)m(\Sigma)$

and

(ii) $U, TU, \cdots, T^{n-1}U$ are disjoint also and $m(U \cup \cdots \cup T^{n-1}U) > 1 - \varepsilon - \delta$.

(2) Let $a = [\exp h(T)] + 1$, $S = \{0, \cdots, a-1\}$. The construction of the finite generator which will be denoted by $\alpha = (U_0, \cdots, U_{a-1})$ is done by induction. For convenience we introduce some notations first. If β is a partition, then for any $s, t \in \mathbf{N}$, $T^{-s}\beta \vee \cdots \vee T^t\beta$ is a partition and will be denoted by $(\beta)_{-s}^t$. $\Lambda(\beta, n, \varepsilon)$ is the union of all "good atoms" of $(\beta)_0^{n-1}$, i.e. $B \in (\beta)_0^{n-1} | \Lambda(\beta, n, \varepsilon)$ satisfies

$$\left| \frac{1}{n} \log m(B) + h(T, \beta) \right| < \varepsilon.$$

Then McMillan's theorem says that $\lim_{n\to\infty} m(\Lambda(\beta, n, \varepsilon)) = 0$.

To begin the construction, let $c \in \mathbf{N}$ such that $h(T) < c^{-1} \log(a^c - 1) =: \log a'$ and choose any refining sequence γ_i $(i = 1, 2, \cdots)$ of finite partitions satisfying

$$\lim_i \max_{C \in \gamma_i} \operatorname{diam} C = 0.$$

(3) In this part we describe the first step in the construction of α.

Let $\delta_1, \varepsilon_1 > 0$ be sufficiently small. If p_1 is large enough then $m(\Lambda(\gamma_1, p_1, \varepsilon_1)) > 1 - \delta_1$. Choosing a Rohlin set for p_1 and δ_1 as in (1), one obtains an open Rohlin set $V_1 \subseteq \Lambda(\gamma_1, p_1, \varepsilon_1)$ for p_1 and $2\delta_1$ with $m(\partial V_1) = 0$. Define the partition β_1 to be $\Omega \setminus \overline{V_1}$ and to be $(\gamma_1)_0^{p_1 - 1}$ on V_1. If δ_1 is small enough then $h(T, \beta_1) \geqq h(T, \gamma_1) - \varepsilon_1$.

Furthermore by definition of $\Lambda(\gamma_1, p_1, \varepsilon_1)$ one has

$$\operatorname{card} \beta_1 \mid V_1 \leqq \exp p_1(h(T, \gamma_1) + \varepsilon_1).$$

Now choose q_1 minimal such that $B \in \beta_1 \mid V_1$ can be coded by a name of length q_1 in S^{q_1} so that every name used has no subblock of zeros of length $\geqq c$ and different B's have different names. We may assume that $q_1 \geqq \frac{1}{2}p_1$, because γ_1 and ε_1 can be chosen to satisfy $h(T, \gamma_1) + \varepsilon_1 \geqq \frac{3}{4}\log a'$. If ε_1 is small enough and p_1 large enough then

$$\frac{q_1 + 2c + 1}{p_1} < \frac{h(T) + 2\varepsilon_1}{\log a'} < 1,$$

and therefore we can define α on

$$\Sigma_1 = \Omega \setminus \bigcup_{t=2c}^{p_1 - q_1 - 2} T^t V_1$$

in the following way. $U_i \cap \Sigma_1$ contains all sets of the form $T^t B$ with $B \in \beta_1 \mid V_1$, $p_1 - q_1 - 1 \leqq t \leqq p_1 - 2$ and where the $(t - p_1 + q_1 + 2)$-th coordinate of the name of B is i. In addition in $U_0 \cap \Sigma_1$ we put the set $\bigcup_{t=0}^{2c-1} T^t V_1$ and in $U_1 \cap \Sigma_1$ the set $\Omega \setminus \bigcup_{t=0}^{p_1-2} T^t V_1$.

A few facts should be noted here. Whatever the partition α will look like on $\Omega \setminus \Sigma_1$ the entrance times into V_1 of a point ω can be decoded very easily: $T^t\omega \in V_1$ iff $T^{t+j}\omega \in U_0$ for every $0 \leqq j < 2c$ and this does not happen at least q_1 times before that. (To see the equivalence use $q_1 \geqq \frac{1}{2}p_1$.) Once knowing the entrance times to V_1 the 1–1 coding of the elements of $\beta_1 \mid V_1$ shows that β_1 is $(\alpha)_{-p_1}^{p_1}$ measurable. Finally we remark that α is defined on all but an open set A_1 of measure

$$(p_1 - q_1 - 2c - 1)m(V_1) \geqq 1 - \frac{h(T) + 2\varepsilon_1}{\log a'}.$$

(4) The induction proceeds similar to the first step and it is therefore sufficient to explain the second one.

Let $\delta_2, \varepsilon_2 > 0$ be sufficiently small. If p_2 is chosen large enough, then $m(\Lambda(\beta_1, p_2, \varepsilon_2)) > 1 - \delta_2$ and $m(\Lambda(\beta_1 \vee \gamma_2, p_2, \varepsilon_2)) > 1 - \delta_2$. Let V_2' be a Rohlin set for p_2, δ_2 and A_1, given by Grillenberger's lemma (see part (1) and take $\varepsilon = \delta = \delta_2$, $V = A_1$). Then for

$$V_2 = V_2' \cap \Lambda(\beta_1, p_2, \varepsilon_2) \cap \Lambda(\beta_1 \vee \gamma_2, p_2, \varepsilon_2)$$

we may assume that $m(V_2) > (1 - 4\delta_2)/p_2$. Again define the partition β_2 to consist of $\Omega \setminus \overline{V_2}$ and to be $(\beta_1 \vee \gamma_2)_0^{p_2-1}$ on V_2, and if δ_2 is small enough it follows that $h(T, \beta_2) \geqq h(T, \gamma_2) - \varepsilon_2$. An easy calculation shows that for $D \in (\beta_1)_0^{p_2-1} \mid V_2$

$$\operatorname{card} \beta_2 \mid D \leqq \exp p_2(h(T) - h(T, \gamma_1) + \varepsilon_1 + 2\varepsilon_2).$$

Now choose q_2 minimal (but $q_2 \geqq \frac{1}{2} p_2 m(A_1)$) such that any $B \in \beta_2 \mid D$ (D fixed) can be given a name in S^{q_2} of length q_2 which does not contain a subblock of more than c zeros and such that different B's have different names. If ε_2 is small enough and p_2 large enough then

$$\frac{q_2 + 2c + 1}{p_2} < \frac{h(T) - h(T, \gamma_1) + \varepsilon_1 + 3\varepsilon_2}{\log a'} < m(A_1),$$

and $k_{p_2} > p_2 m(A_1)$ shows that $q_2 + 2c + 1 < k_{p_2}$. Therefore α can be defined on $\Sigma_2 = A_1 \setminus \bigcup_{t=2c}^{k_{p_2}-q_2-2} T_{A_1}^t V_2$ in the same way as in the first step using T_{A_1} instead of T.

Exactly as in (3) the entrance times into V_2 can be decoded (because those into V_1 are known and therefore T_{A_1} is known). Furthermore since the partition β_1 can be determined from the first step the 1–1 coding of the elements of β_2 on the atoms of $(\beta_1)_0^{p_2-1} \mid V_2$ shows again that β_2 is $(\alpha)_{-p_2}^{p_2}$ measurable, whatever the partition α will be on the remaining part $A_2 = A_1 \setminus \overline{\Sigma_2}$. Finally one observes that

$$m(A_2) = (k_{p_2} - q_2 - 2c - 1) m(V_2) \geqq 1 - \frac{h(T) + \varepsilon_1 + 3\varepsilon_2}{\log a'}.$$

(5) Proceeding in this way we obtain open sets U_i. In order to see that (U_i) is a partition note that in every step of the construction at least half of the remaining set is partitioned.

It is left to show that α is a generator. Define as in the proof of Theorem 18

$$\Omega_2 := \bigcap_{n \geqq 1} \bigcup_{i \geqq n} \bigcup_{t=0}^{p_i - 1} T^t V_i.$$

Clearly Ω_2 is a residual set of full measure and therefore the property (iv) of Proposition 17 has to be checked on Ω_2 for showing α to be a generator. But this is easily done as in Theorem 18 using what is said about the decoding of the entrance times of the V_i's and the sets in $\beta_i \mid V_i$.

It is possible to obtain generators satisfying additional conditions, such as having images in a given mixing Markov shift of strictly larger entropy, (see [9], prop. 283, p. 295), and it might be useful to generalize the generator constructions in [9], theor. 30.1, p. 309 and [14] to almost topological dynamical systems. Another question would be whether a finitary embedding in any ergodic dynamical system in the two-dimensional torus (with a homeomorphism) is possible (see [24] for the metric case).

§4. Finite expected coding times

Let $(X, \mathcal{A}, \mu, S)$ and $(Y, \mathcal{B}, \nu, S)$ be shift dynamical systems, and let φ be a finitary homomorphism from X to Y. Such a homomorphism is nothing other than a *sequential coding procedure*. That is, if we want to find the image $\varphi x = y$ of a point $x \in X$ under φ, we look at the coordinates $x_{-n}, x_{-n+1}, \cdots, x_0, \cdots, x_n$ of x for successive $n = 0, 1, 2, \cdots$, until we find such a sequence that the corresponding cylinder set

$$\{z \in X : z_k = x_k \text{ for } |k| \leqq n\}$$

is contained in $\varphi^{-1}(\{w \in Y : w_0 = i\})$ for some i. Since this latter set is a countable union of cylinders, this will happen at a finite index n for almost all $x \in X$. Then we can set $y_0 = i$. By shifting and repeating the procedure, all the y_k can be successively determined. Probabilistically, this means that there is a stopping time defined on X such that the zero coordinate of the image of a point is determined by the stopped sequence. The idea of this paragraph is that if the stopping time has finite expectation, then certain properties are preserved under φ.

Definition 21. Let f be a bounded measurable function on the shift dynamical system $(X, \mathcal{A}, \mu, T)$, and denote by $\mathcal{A}(-k, k)$ the σ-algebra generated by the coordinate mappings $x \to x_n$ with $|n| \leqq k$. We say that f is *sequential* if there exists a sequence of bounded functions f_k such that:

(i) for each $k \geqq 0$, f_k is $\mathcal{A}(-k, +k)$-measurable,

(ii) $\lim_{k\to\infty} \int |f - \sum_{n=0}^{k} f_k| \, d\mu = 0$, and

(iii) $\sum_{k=0}^{\infty} k \int |f_k| \, d\mu < \infty$.

Let $\mathcal{S}(X) = \{f : f \text{ is sequential}\}$, and for $f \in \mathcal{S}(X)$ let $\sigma(f)$ denote the infinum over all possible values in (iii) for different f_k.

Note that if U is an open set in X, then 1_U is sequential iff $U = \sum_{k=0}^{\infty} C_k$, $C_k \in \mathcal{A}(-k, +k)$, and $\sum_{k=0}^{\infty} k\mu(C_k) < \infty$, and the C_k can be chosen such that

$$\sigma(U) = \sigma(1_U) = \sum_{k=0}^{\infty} k\mu(C_k).$$

DEFINITION 22. Let φ be a finitary homomorphism from $(X, \mathcal{A}, \mu, S)$ to $(Y, \mathcal{B}, \nu, S)$. If g is a bounded measurable function on Y, then we say that φ *codes g sequentially* if $g \circ \varphi \in \mathcal{S}(X)$. The homomorphism φ is said to have *finite expectation* if there exists a constant K such that for any set $C \subseteq Y$ with $C \in \mathcal{B}(-k, +k)$, φ codes 1_C sequentially and

$$\sigma(1_C \circ \varphi) \leqq Kk\nu(C).$$

THEOREM 23. *If φ has finite expectation, then φ codes g sequentially for any $g \in \mathcal{S}(Y)$, i.e. $\varphi(\mathcal{S}(Y)) \subseteq \mathcal{S}(X)$.*

PROOF. Let $g = \Sigma g_k$ with g_k $\mathcal{B}(-k, k)$ measurable and

$$\sum_{k=0}^{\infty} k \int |g_k| \, d\nu < \infty.$$

Then $g \circ \varphi = \Sigma_{k=0}^{\infty} g_k \circ \varphi$, and if $g_k = \Sigma_j \alpha_j^{k} 1_{C_j^k}$, where C_j^k are $\mathcal{B}(-k, k)$ measurable and disjoint, we have

$$g \circ \varphi = \sum_{k=0}^{\infty} \sum_j \alpha_j^k 1_{\varphi^{-1}(C_j^k)}.$$

Now choose for each j and k sets $C_{jn}^k \in \mathcal{A}(-n, n)$ such that

$$\varphi^{-1}(C_j^k) = \sum_{n=0}^{\infty} C_{jn}^k$$

and

$$\sigma(1_{\varphi^{-1}(C_j^k)}) = \sum_{n=0}^{\infty} n\nu(C_{jn}^k) \leqq Kk\nu(C_j^k).$$

Then

$$g \circ \varphi = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \sum_j \alpha_j^k 1_{C_{jn}^k} \right) = \sum_{n=0}^{\infty} f_n,$$

the functions f_n are $\mathcal{A}(-n, n)$-measurable, and

$$\begin{aligned} \sum_{n=0}^{\infty} n \int |f_n| \, d\mu &\leqq \sum_{n=0}^{\infty} n \sum_{k=0}^{\infty} \sum_j |\alpha_j^k| \mu(C_{jn}^k) \\ &\leqq \sum_{k=0}^{\infty} \sum_j Kk |\alpha_j^k| \nu(C_j^k) \\ &= K \sum_{k=0}^{\infty} k \int |g_k| \, d\nu < \infty. \end{aligned}$$

To formulate our next result, we recall that the dynamical system $(X, \mathscr{A}, \mu, T)$ is said to be *α-mixing* (or strongly mixing) if

$$\alpha(t) = \sup_{\substack{A \in \mathscr{A}(-\infty,\tau) \\ B \in \mathscr{A}(t+\tau,\infty)}} |\mu(A \cap B) - \mu(A)\mu(B)| \to 0$$

as $t \to \infty$. The function $\alpha(t)$ is called the mixing coefficient of the system.

THEOREM 24. *Let $(X, \mathscr{A}, \mu, S)$ and $(Y, \mathscr{B}, \nu, S)$ be shift dynamical systems, and let φ be a finitary homomorphism from X to Y. If*

(a) $(X, \mathscr{A}, \mu, S)$ is α-mixing,

(b) $\sum_{t=1}^{\infty} \alpha(t) < \infty$, and

(c) φ has finite expectation,

then for any $g \in \mathscr{S}(Y)$ the sum

$$\sigma^2 = \int (g - \int g d\nu)^2 d\nu + 2 \sum_{t=1}^{\infty} \int (g - \int g d\nu)(S^t g - \int g d\nu) d\nu$$

converges, and if $\sigma^2 \neq 0$, then

$$\lim_{n \to \infty} \nu\Big(\frac{1}{\sigma\sqrt{n}} \sum_{t=0}^{n-1} S^t g - \int g d\nu < z\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2} du.$$

PROOF. Let $f = g \circ \varphi$. Then the processes $(S^t f)$ and $(S^t g)$ have the same joint distributions (since φ preserves measure), and $f \in \mathscr{S}(X)$. According to [15], theor. 18.6.3, p. 356, it therefore suffices to show that for any $f \in \mathscr{S}(X)$,

$$(*) \qquad \sum_{n=0}^{\infty} \int |f - E(f \mid \mathscr{A}(-n, n))| d\mu < +\infty.$$

Let $f = \sum_{k=0}^{\infty} f_k$ as in Definition 21. Then

$$E(f \mid \mathscr{A}(-n, n)) = \sum_{k=0}^{n} f_k + E\Big(\sum_{k>n} f_k \mid \mathscr{A}(-n, n)\Big)$$

and

$$\int |f - E(f \mid \mathscr{A}(-n, n))| d\mu = \int \Big| \sum_{k>n} f_k - E\Big(\sum_{k>n} f_k \mid \mathscr{A}(-n, n)\Big)\Big| d\mu$$
$$\leqq 2 \sum_{k>n} \int |f_k| d\mu.$$

Thus the sum in $(*)$ is bounded by

$$2\sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}\int |f_k|\,d\mu = 2\sum_{n=1}^{\infty} n\int |f_n|\,d\mu < \infty,$$

since $f \in \mathcal{S}(X)$, and the theorem is proved.

A more general result is certainly possible (see [12] and [15] theor. 18.6.1 and 18.6.2), but we have chosen a simple formulation. If the central limit theorem is required only for functions on Y which are measurable with respect to a finite number of coordinates, the condition that φ have finite expectation can be weakened, and we need only require that the cylinders of length one in Y be sequentially coded by φ (this implies that all $\mathcal{B}(-n, n)$-measurable functions are sequentially coded).

It is easy to show that all "sufficiently smooth" functions on Y belong to $\mathcal{S}(Y)$. That is, let g be a function on Y and set for each $n \geqq 0$

$$\delta(g, n) = \sup_{C_n} \sup_{x,y\in C_n} |g(x) - g(y)|,$$

where the first supremum is taken over all cylinder sets of length $2n+1$ in $\mathcal{B}(-n, +n)$. We say that g is *sufficiently smooth* if $\sum_{n=0}^{\infty} n\delta(g, n) < \infty$.

THEOREM 25. *If g is sufficiently smooth, then $g \in \mathcal{S}(Y)$.*

PROOF. Let h_k denote the conditional expectation of g with respect to the σ-algebra $\mathcal{B}(-k, k)$. If C_k is a cylinder in $\mathcal{B}(-k, k)$ and if $x \in C_k$, then

$$|g(x) - h_k(x)| = \left| \int_{C_k} \frac{g(x) - g(y)}{\nu(C_k)}\, d\nu(y) \right| \leqq \delta(g, k).$$

We set $g_0 = h_0$, $g_k = h_k - h_{k-1}$ ($k \geqq 1$). Then $g_0 + g_1 + \cdots + g_k = h_k$ and since our hypothesis implies that $\delta(g, k) \to 0$ as $k \to \infty$, we have

$$g = \sum_{k=0}^{\infty} g_k.$$

Moreover, g_k is $\mathcal{B}(-k, k)$-measurable, and

$$\sum_{k=0}^{\infty} k\int |g_k|\,d\nu \leqq 2\sum_{k=0}^{\infty} k\int |g - h_{k-1}|\,d\nu \leqq 2\sum_{k=0}^{\infty} k\delta(g, k-1) < \infty$$

because g is sufficiently smooth. Hence $g \in \mathcal{S}(Y)$.

COROLLARY 26. *Let $(Y, \mathcal{B}, \nu, S)$ be an almost topological dynamical system which is finitarily isomorphic to (or a finitary homomorphic image of) a Bernoulli* *scheme, and suppose that the finitary homomorphism from the Bernoulli scheme to Y has finite expectation. Then any sufficiently smooth function on Y satisfies the central limit theorem, in the sense of Theorem* 24.

We conclude with some problems which seem to be worthwhile.

(1) Does there exist an ergodic dynamical system of zero entropy and a function on this system which satisfies the central limit theorem? (This problem is due to J. P. Conze.)

(2) It is known (see [20]) that if X and Y are Bernoulli schemes with the entropy of X strictly larger than that of Y, then there exists a finitary homomorphism from X to Y, and it is not hard to see that the homomorphism of [20] has finite expectation. The interesting and still open problem of whether two Bernoulli schemes of the same entropy are finitarily isomorphic merits attention.[†]

(3) Different classical systems have been recently shown to be metrically isomorphic to Bernoulli schemes, and if one could obtain finitary isomorphisms with finite expectations for such systems (they are in general not defined on shift spaces, but it is easy to extend our definition, say, relative to a given generator) Corollary 26 would provide solid evidence that physical measurements on such classical systems yield random data.

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[†] *Note added in proof.* It has recently been proved that two Bernoulli schemes of the same entropy are finitarily isomorphic (M. Keane and M. Smorodinsky, *Bernoulli schemes of the same entropy are finitarily isomorphic*, to appear in Ann. Math. **109** (1979)), and even that two mixing Markov shifts of the same entropy are finitarily isomorphic (M. Keane and M. Smorodinsky, *The finitary isomorphism theorem for Markov shifts*, Bull. Amer. Math. Soc., March 1979). These isomorphisms have been shown to have infinite expected coding times in general (W. Parry, *Finitary isomorphisms with finite expected code lengths*, preprint, Univ. of Warwick, July 1978).

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